

# On the Physical Dynamics of Laser Mode Sweeping in a Fabry-Perot Resonator of Varying Optical Length

F. Bodem

Institut für Hochfrequenztechnik of the University of Erlangen-Nürnberg

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We present in this paper a basic analysis of the physical mechanism underlying the internal frequency modulation of a laser oscillator which is caused by a variation of the optical path length of its Fabry-Perot resonator. It is shown how adiabatic variations of the optical resonator length are continuously converted into frequency variations of the electromagnetic field energy trapped between the resonator mirrors by a rapid succession of small Doppler shifts, thus complying with the general principle of adiabatic invariance in oscillatory systems.

## 1. Introduction

The frequency sweep of the output signal of a laser oscillator due to a variation of the optical length of its Fabry-Perot resonator is a well-known phenomenon [1, 2]. In most cases this effect is considered an origin of disturbing or at least unnecessary instabilities both of the frequency and the amplitude of the laser output signal with a corresponding limitation of its coherence length. In some cases, on the other hand, laser mode sweeping has been generated systematically in context with attempts to control the spiking behaviour of solid state lasers [3–7]. Moreover, mode sweeping of a single-mode laser oscillator with a wide gain profile of its active medium is the only practical means of generating a frequency modulated light signal with a considerable frequency swing and may, therefore, be of some interest for certain optical experiments.

In the literature on mode sweeping, the variation of the laser output frequency is usually considered to be the result of a continuous shift of the stationary resonant frequencies of the Fabry-Perot resonator due to a continuous change of the optical path length between its mirrors. The mathematical description given in this context most often reads as follows: The resonant frequency  $f_q$  of a TEM-mode is approximately given by [8]

$$f_q = q c_0 / 2l, \quad (1)$$

where  $c_0$  is the speed of light in vacuo,  $l$  is the optical length of the Fabry-Perot resonator, and  $q$  is the number of half wavelengths of the standing light wave trapped between its mirrors. Differen-

tiation of Eq. (1) with respect to time yields

$$\dot{f}_q = -q(c_0/2l^2)\dot{l} = -f_q v/l, \quad (2)$$

where  $v = \dot{l}$  is the speed of a change in optical path length, which may be induced by a moving mirror or the changing refractive index of a section of the Fabry-Perot resonator, for instance. Although Eq. (2) has been obtained in a formally correct way and is in keeping with the experimentally observed effects, its derivation gives no account of the physical mechanism which actually changes the frequency of the electromagnetic energy enclosed within the resonator. We therefore present in the following in a simplified model the basic physics of internal laser frequency modulation by mode sweeping.

## 2. The Physical Mechanism of Linear Mode Sweeping

We consider first the laser frequency sweep generated by a small linear axial displacement at constant velocity of one of the mirrors of a Fabry-Perot resonator (Figure 1). It will be assumed

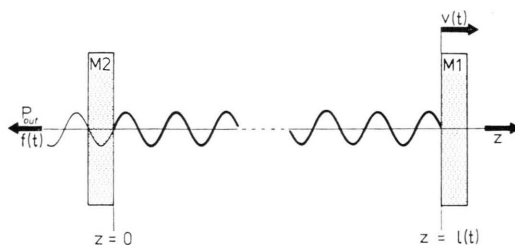


Fig. 1. A linearly polarized standing light wave of frequency  $f_q$  as a resonant mode of a Fabry-Perot resonator. The totally reflecting mirror M1 is assumed to start at  $t = 0$  a small parallel displacement at velocity  $v = \dot{l}$ , resulting in an optical resonator length  $l(t) = l_0 + \int_0^t v dt$ . The partially transmitting mirror M2 generates the laser output signal ( $P_{out}, f(t)$ ).

Reprint requests to Dr. Friedrich Bodem, Institut für Hochfrequenztechnik der Universität Erlangen, Cauerstr. 9 D-8520 Erlangen.



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throughout the following that the frequency deviations obtained will always stay within the net gain profile of the active laser medium occupying the resonator, thus ensuring that laser operation is being maintained in every instant. We further assume frequency pulling effects caused by the resonant interaction between the resonator mode and the laser medium [9] to be negligible as compared to the passive frequency sweep caused by the resonator.

The transverse electric field  $E(x, y, z, t)$  of a linearly polarized TEM-mode of the resonator shown in Fig. 1 may be decomposed into its constituent waves  $E^+(x, y, z, t)$  and  $E^-(x, y, z, t)$  traveling against each other along the  $z$ -axis:

$$E(x, y, z, t) = E^+(x, y, z, t) + E^-(x, y, z, t), \quad (3)$$

where

$$E^+(x, y, z, t) = \hat{E}^+(x, y, z) \cdot \exp \{j 2\pi f_q(t - z/c)\}, \quad (4)$$

$$E^-(x, y, z, t) = \hat{E}^-(x, y, z) \cdot \exp \{j 2\pi f_q(t + z/c)\}. \quad (5)$$

$\hat{E}^+(x, y, z)$  and  $\hat{E}^-(x, y, z)$  are spatial structure functions [8] of the given mode,  $c$  is the speed of light within the resonator, and  $f_q$  is the mode frequency at  $t=0$ . We consider now the effect of a small parallel displacement at constant velocity  $v$  of mirror M1 which starts at  $t=0$ . The traveling wave  $E^-(x, y, z, t)$  reflected at mirror M1 will

exhibit a Doppler frequency shift  $\Delta f$  as compared to  $E^+(x, y, z, t)$ , which in the nonrelativistic case ( $v \ll c_0$ ) is [10]

$$\Delta f \cong f_q 2v/c_0. \quad (6)$$

Additional Doppler shifts will occur after multiples of the resonator round trip time

$$\tau = 2l(t)/c_0 \cong 2l_0/c_0. \quad (7)$$

The thus resulting temporal behaviour of the frequency  $f(t)$  of the laser output signal transmitted by mirror M2 is shown in Figure 2.

For  $t \gg \tau$ ,  $v \ll c_0$ , and  $vt \ll l_0$  we may write:

$$f(t) \cong f_q \left(1 - \frac{2v}{c_0}\right)^{t/\tau} \cong f_q \left(1 - \frac{2vt}{c_0\tau}\right) \cong f_q \left(1 - \frac{vt}{l_0}\right) \quad (8)$$

and

$$\dot{f}(t) \cong -f_q v/l_0. \quad (9)$$

Interesting enough, Eq. (9) is in keeping with Eq. (2) although it has been obtained in a completely different (and physically correct) manner. We can further show that the frequency  $f(t)$  resulting from Eq. (8) for the light wave circulating in the resonator is in every instant a momentary resonant frequency, as

$$f(t) \cong f_q \left(1 - \frac{vt}{l_0}\right) \cong \frac{qc_0}{2l_0(1 + vt/l_0)} = q \frac{c_0}{2l(t)}. \quad (10)$$

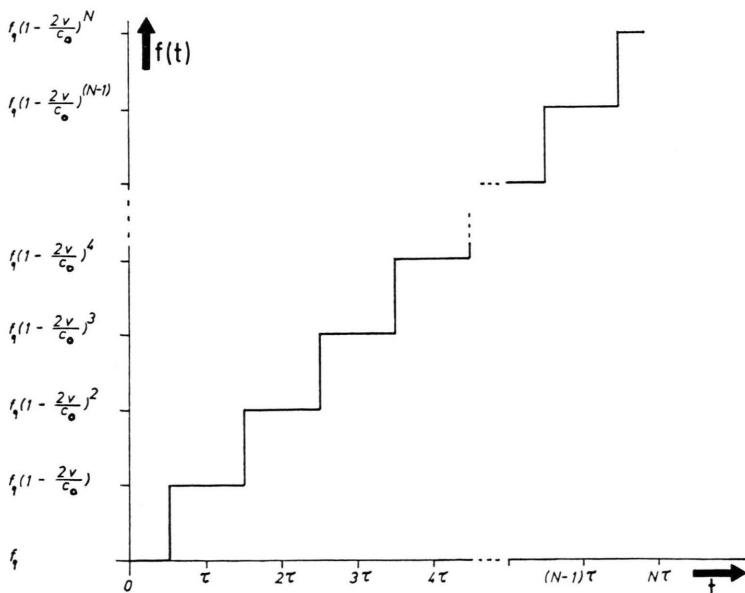


Fig. 2. Temporal behaviour of the frequency  $f(t)$  of the laser output signal transmitted by mirror M2 (Fig. 1) during a small parallel displacement at constant velocity  $v$  of mirror M1 starting at  $t=0$ .

Thus, if the motion of mirror M1 is stopped, the trapped light wave is instantly a stationary resonant mode of the Fabry-Perot resonator.

The underlying physical principle of this fact, which applies to the following considerations as well, is the well-known adiabatic invariance of the ratio of the total energy to the frequency of an oscillatory system [11], as an adiabatic motion of mirror M1 is accompanied by a small amount of energy given to or delivered by the resonator via its pressure of radiation [1]. The Fabry-Perot resonator behaves in a formally similar way as, for instance, a swinging pendulum, the length of which is slowly changed in its point of suspension thus changing both its frequency and energy. We may note in passing that a further development of this principle in the present case would suggest in the well-known way [12] the quantized photon structure of the electromagnetic field energy stored in the resonator.

Equation (8) is obtained in quite the same manner in cases where the variation in optical path length is generated by mechanisms other than a moving mirror. If a section of length  $d$  of the resonator is occupied by a medium of refractive index  $n$ , a variation  $d$  and/or  $n$  causes a frequency shift

$$\Delta f \cong f_q 2\dot{l}/c_0 = f_q (2/c_0) (\dot{n} d + \dot{d} (n-1)) \quad (11)$$

of the circulating light wave during every round trip time  $\tau$ , which yields the same effect as the Doppler shift considered above.

### 3. Laser Frequency Modulation by Arbitrary Signals

We are now prepared to analyze the physics of an internal laser frequency modulation by an arbitrary signal  $s(t)$  which is converted into small proportional variations of the optical resonator length:

$$l(t) = l_0 + l_1(t) = l_0 + A s(t), \quad (12)$$

where  $A$  is a constant and the modulation is assumed to start at  $t=0$  and  $l_1(t)=0$ . We express  $l_1(t)$  by the Fourier integral

$$l_1(t) = \int_0^\infty a(f_m) \cos(2\pi f_m t + \varphi(f_m)) df_m, \quad (13)$$

where  $a(f_m)$  and  $\varphi(f_m)$  are real functions. For the corresponding sweep velocity of the optical path length we obtain

$$v(t) = -2\pi \int_0^\infty a(f_m) f_m \sin(2\pi f_m t + \varphi(f_m)) df_m. \quad (14)$$

For the frequency  $f(t)$  of the laser output signal due to repeatedly occurring Doppler frequency shifts  $\Delta f$  we obtain in a similar manner as in the preceding section

$$\begin{aligned} f(t) &= f_q \prod_{k=1}^N \left( 1 - \frac{2v(k\tau)}{c_0} \right) \\ &= f_q \prod_{k=1}^N \left( 1 + \frac{4\pi}{c_0} \int_0^\infty a(f_m) f_m \sin(2\pi f_m k\tau + \varphi(f_m)) df_m \right), \end{aligned} \quad (15)$$

where  $N \cong t/\tau$ . The product in Eq. (15) can be evaluated to yield (see appendix)

$$f(t) \cong f_q \left( 1 - \frac{l_1(t)}{l_0} \right). \quad (16)$$

We thus obtain within the limits set by the approximations used in the derivation of this result the expected linear frequency modulation of the laser output signal.

Again, the frequency  $f(t)$  obtained from Eq. (16) is in every instant a momentary resonant frequency of the Fabry-Perot resonator, as we may write ( $|l(t)| \ll l_0$ )

$$\begin{aligned} f(t) &\cong \frac{q c_0}{2l_0} \left( 1 - \frac{l(t)}{l_0} \right) \\ &\cong \frac{q c_0}{2l_0(1 + (l(t)/l_0))} = \frac{q c_0}{2l(t)}. \end{aligned} \quad (17)$$

Hence, whenever an arbitrary adiabatic variation of optical path length of the Fabry-Perot resonator is stopped, the electromagnetic energy stored in it instantly occupies a stationary resonant mode.

### 4. Conclusion

We have tried to derive in a physically correct way a basic description of the frequency modulation effect observed in laser oscillators with a Fabry-Perot resonator of varying optical length. It could be shown that in the practically important adiabatic case mode sweeping has to be considered as a continuous frequency pulling of one stationary laser mode and not as a consecutive dissociate buildup of laser modes at different resonant frequencies of the resonator, as the most often encountered formal derivation in Eq. (2) might perhaps suggest.

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## Appendix

### Evaluation of the Product in Eq. (15)

Taking the logarithm of both sides yields

$$\ln f(t) = \ln f_q + \sum_{k=1}^N \ln \left( 1 - \frac{2v(k\tau)}{c_0} \right) \quad (\text{A.1})$$

As  $v \ll c_0$  has been assumed, the logarithms in (A.1) may be approximated to yield

$$\begin{aligned} \ln f(t) &= \ln f_q + \sum_{k=1}^N \left( -\frac{2v(t)}{c_0} \right) \\ &= \ln f_q + \sum_{k=1}^N \frac{4\pi}{c_0} \int_0^\infty a(f_m) f_m \sin(2\pi f_m k\tau + \varphi(f_m)) df_m \\ &= \ln f_q + \frac{4\pi}{c_0} \int_0^\infty a(f_m) f_m \left( \sum_{k=1}^N \sin(2\pi f_m k\tau + \varphi(f_m)) \right) df_m. \end{aligned} \quad (\text{A.2})$$

The summation of the Fourier series in (A.2) yields

$$\begin{aligned} &\sum_{k=1}^N \sin(2\pi f_m k\tau + \varphi(f_m)) \\ &= \frac{\sin(2\pi f_m N\tau + \varphi(f_m)) + \sin(2\pi f_m \tau + \varphi(f_m)) - \sin(2\pi f_m (N+1)\tau + \varphi(f_m))}{2 - 2\cos 2\pi f_m \tau} \\ &\cong -\frac{\cos(2\pi f_m \tau + \varphi(f_m)) - \cos \varphi(f_m)}{2\pi f_m \tau} \end{aligned} \quad (\text{A.3})$$

where  $f_m \tau \ll 1$  and  $N\tau \cong t$  have been used in the last approximation. With (A.2) and (A.3) we obtain

$$\begin{aligned} \ln f(t) &\cong \ln f_q - \frac{1}{l_0} \left( \int_0^\infty a(f_m) \cos(2\pi f_m t + \varphi(f_m)) df_m - \int_0^\infty a(f_m) \cos \varphi(f_m) df_m \right) \\ &= \ln f_q - \frac{1}{l_0} (l_1(t) - l_1(t=0)). \end{aligned} \quad (\text{A.4})$$

and finally with  $l_1(t=0) = 0$  and  $|l_1(t)| \ll l_0$

$$f(t) \cong f_q \exp \left( -\frac{l_1(t)}{l_0} \right) \cong f_q \left( 1 - \frac{l_1(t)}{l_0} \right). \quad (\text{A.5})$$

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